THE LANGUAGE OF THEOREMS

Many theorems in math are worded using "if … then" sentences.

For example, a theorem (let's call it "the M5 theorem") might say

> "If the remainder when x is divided by 5 is 1
> and the remainder when y is divided by 5 is 2,
> then $x^2 + y^2$ is a multiple of 5."

The way you use a theorem is that
if you can show that the "if" part of the theorem is true in a certain situation,
you immediately know the "then" part is true without doing any other work.

For example, suppose $x = 9371$ and $y = 8562$.
Since the remainder when $x = 9371$ is divided by 5 is 1
and the remainder when $y = 8562$ is divided by 5 is 2,
then without doing anything else, because of the M5 theorem, you know $x^2 + y^2$ is a multiple of 5.
No squaring, no adding and no additional dividing are required.

On the other hand, suppose $x = 6074$ and $y = 2412$.
Since the remainder when $x = 6074$ is divided by 5 is **not** 1,
(even though the remainder when $y = 2412$ is divided by 5 is 2,)
the M5 theorem doesn't tell us anything about whether $x^2 + y^2$ is a multiple of 5.
$x^2 + y^2$ might still be a multiple of 5, or it might not, but the M5 theorem can't tell you which.
You will have to do some other work to figure it out. (By the way, $x^2 + y^2$ is a multiple of 5 in this case.)

To understand theorems, you have to understand

> what is meant by "the conditions and conclusion of a theorem"
> what it means to "satisfy the conditions of a theorem" or for "a theorem to apply"
> what it means to "satisfy the conclusion of a theorem"
> what it means to "contradict (or not contradict) an if/then statement"

The "if" part of a theorem is called the **conditions**, **assumptions** or **hypotheses** of the theorem.

The **conditions** of the M5 theorem above are

> the remainder when x is divided by 5 is 1
> the remainder when y is divided by 5 is 2

A situation **satisfies the conditions** of a theorem if, in that situation, all the conditions are true.
Another way of saying "a situation **satisfies the conditions** of a theorem" is "**the theorem applies** to the situation".

> In order to show that a situation **satisfies the conditions** of a theorem
> (or **the theorem applies** to the situation**),**
> you must show that all the conditions are true.

> In order to show that a situation **does not satisfy the conditions** of a theorem
> (or **the theorem does not apply** to the situation**),**
> you must show that one of the conditions is false.

SITUATION ⬇ ⬇	satisfies the conditions of the M5 theorem (the M5 theorem applies)	**because**	remainder when x is divided by 5 is 1	**and**	remainder when y is divided by 5 is 2
$x=1,\ y=2$	YES		**YES**		**YES**
$x=1,\ y=3$	NO		YES		**NO**
$x=4,\ y=2$	NO		**NO**		YES
$x=4,\ y=3$	NO		**NO**		**NO**
$x=1,\ y=4$	NO		YES		**NO**
$x=3,\ y=2$	NO		**NO**		YES
$x=0,\ y=3$	NO		**NO**		**NO**

NOTE: **In the four cases** $x=1,\ y=3$ **or** $x=4,\ y=2$ **or** $x=1,\ y=4$ **or** $x=3,\ y=2$
one of the conditions is satisfied. However, because the other condition is not satisfied,
we say the conditions are not satisfied.

The "then" part of a theorem is called the **conclusion** of the theorem.

The **conclusion** of the M5 theorem is

$x^2 + y^2$ is a multiple of 5

A situation **satisfies the conclusion** of a theorem if, in that situation, the conclusion is true.

In order to show that a situation **satisfies the conclusion** of a theorem,
you must show that the conclusion is true.

In order to show that a situation **does not satisfy the conclusion** of a theorem,
you must show that the conclusion is false.

NOTE: **If you are only trying to determine whether a situation satisfies the conclusion of a theorem, you do NOT consider whether the conditions are satisfied.**

SITUATION ⬇ ⬇	satisfies the conclusion of the M5 theorem	**because**	$x^2 + y^2$ is a multiple of 5
$x = 1,\ y = 2$	YES		YES $(1^2 + 2^2 = 5)$
$x = 1,\ y = 3$	YES		YES $(1^2 + 3^2 = 10)$
$x = 4,\ y = 2$	YES		YES $(4^2 + 2^2 = 20)$
$x = 4,\ y = 3$	YES		YES $(4^2 + 3^2 = 25)$
$x = 1,\ y = 4$	NO		NO $(1^2 + 4^2 = 17)$
$x = 3,\ y = 2$	NO		NO $(3^2 + 2^2 = 13)$
$x = 0,\ y = 3$	NO		NO $(0^2 + 3^2 = 9)$

NOTE: **In the three cases** $x = 1,\ y = 3$ **or** $x = 4,\ y = 2$ **or** $x = 4,\ y = 3$
the conditions are not satisfied. That doesn't change the fact that the conclusion is satisfied.

A situation **contradicts an if/then statement** if the situation **satisfies the conditions** of the statement, but **does not satisfy the conclusion** of the statement.

> In order to show that a situation **contradicts an if/then statement**,
> you must show that the situation **satisfies the conditions (the statement applies),**
> but **does not satisfy the conclusion** of the statement.
> That is, you must show that all the conditions are true, but the conclusion is false.
>
> In order to show that a situation **does not contradict an if/then statement**,
> either you must show that the situation **does not satisfy the conditions (the statement does not apply)**
> or the situation **satisfies the conditions (the statement applies) and the conclusion** of the statement.
> That is, either you must show that one of the conditions is false,
> or you must show that the conditions and the conclusion are true.

	contradicts the M5 theorem	satisfies the conditions of the M5 theorem	satisfies the conclusion of the M5 theorem
$x = 1,\ y = 2$	NO	**YES**	**YES**
$x = 1,\ y = 3$	NO	**NO**	YES
$x = 4,\ y = 2$	NO	**NO**	YES
$x = 4,\ y = 3$	NO	**NO**	YES
$x = 1,\ y = 4$	NO	**NO**	NO
$x = 3,\ y = 2$	NO	**NO**	NO
$x = 0,\ y = 3$	NO	**NO**	NO

NOTE: **In the three cases** $x = 1,\ y = 3$ **or** $x = 4,\ y = 2$ **or** $x = 4,\ y = 3$ **the conclusion is satisfied, even though the conditions are not satisfied. That does not contradict the M5 theorem, because the theorem only says what must happen in situations which satisfy the conditions, and the theorem says nothing about what might or might not happen in situations that do not satisfy the conditions.**

You may have noticed that none of the examples contradict the M5 theorem. And you won't ever be able to find an example which contradicts the M5 theorem. That's because, in math, an if/then statement is not called a theorem if it is possible to contradict it. If you can contradict an if/then statement, in math, we simply say the statement is false.

(NOTE: The word "theorem" is not the same as the word "theory", which is used in other branches of science (eg. theory of relativity, theory of evolution).)

Some problems for you to try, based on the following theorem, which we will call the M60 theorem:

> “If x is a multiple of 15
> and y is a multiple of 4,
> then xy is a multiple of 60.”

[1] What are the conditions of the M60 Theorem ? HINT: There are 2 of them.

[2] What is the conclusion of the M60 Theorem ?

[3] Which of the following situations do NOT satisfy the conditions of the M60 Theorem ?
Name all conditions which are NOT satisfied.

- [a] $x = 30$ and $y = 10$
- [b] $x = 9$ and $y = 30$
- [c] $x = 9$ and $y = 40$
- [d] $x = 20$ and $y = 16$
- [e] $x = 30$ and $y = 8$
- [f] $x = 45$ and $y = 6$
- [g] $x = 18$ and $y = 30$

[4] Which of the situations in [3] do NOT satisfy the conclusion of the M60 Theorem ?

[5] Why does each of the situations in [3] NOT contradict the M60 Theorem ?
Answer in terms of conditions and conclusions. Don’t just answer “because it’s a theorem”.

[6] Is it possible to find a function and a domain which contradict the M60 Theorem ?
Why or why not ?

[1] What are the conditions of the M60 Theorem ? HINT: There are 2 of them.

x is a multiple of 15, and
y is a multiple of 4

[2] What is the conclusion of the M60 Theorem ?

xy is a multiple of 60

[3] Which of the following situations do NOT satisfy the conditions of the M60 Theorem ? Name all conditions which are NOT satisfied.

[a] $y = 10$ is not a multiple of 4
[b] $x = 9$ is not a multiple of 15 and $y = 30$ is not a multiple of 4
[c] $x = 9$ is not a multiple of 15
[d] $x = 20$ is not a multiple of 15
[e] satisfies the conditions
[f] $y = 6$ is not a multiple of 4
[g] $x = 18$ is not a multiple of 15 and $y = 30$ is not a multiple of 4

[4] Which of the situations in [3] do NOT satisfy the conclusion of the M60 Theorem ?

[a] satisfies the conclusion since $xy = 300$ which is a multiple of 60
[b] does not satisfy the conclusion since $xy = 270$ which is not a multiple of 60
[c] satisfies the conclusion since $xy = 360$ which is a multiple of 60
[d] does not satisfy the conclusion since $xy = 320$ which is not a multiple of 60
[e] satisfies the conclusion since $xy = 240$ which is a multiple of 60
[f] does not satisfy the conclusion since $xy = 270$ which is not a multiple of 60
[g] satisfies the conclusion since $xy = 540$ which is a multiple of 60

[5] Why does each of the situations in [3] NOT contradict the M60 Theorem ?
Answer in terms of conditions and conclusions. Don’t just answer “because it’s a theorem”.

[e] satisfies both the conditions and the conclusion, so it does not contradict the theorem
[a], [c], [g] do not satisfy the conditions, so the theorem does not apply, so they do not contradict the theorem (it is irrelevant that they satisfy the conclusion)
[b], [d], [f] do not satisfy the conditions, so the theorem does not apply, so they do not contradict the theorem (it is irrelevant that they do not satisfy the conclusion)

[6] Is it possible to find a function and a domain which contradict the M60 Theorem ?
Why or why not ?

No. Because the M60 Theorem is a theorem, which means you cannot find any situation which contradicts it.

Some problems for you to try after we cover section 2.5, based on the Intemediate Value Theorem (Math 1A):

[1] What are the conditions of the Intermediate Value Theorem ? HINT: There are 2 of them.
[2] What is the conclusion of the Intermediate Value Theorem ?
[3] Which of the following situations do NOT satisfy the conditions of the Intermediate Value Theorem ? Name all conditions which are NOT satisfied.

[a] $f(x) = 2x + 3$ on $[1, 4]$ with $d = 9$
[b] $f(x) = x^2$ on $[-3, 4]$ with $d = 5$
[c] $f(x) = \frac{1}{x^2}$ on $[-2, 1]$ with $d = \frac{1}{2}$
[d] $f(x) = \frac{1}{x^2}$ on $[-2, 1]$ with $d = 4$
[e] $f(x) = x^2$ on $[-3, 4]$ with $d = 25$
[f] $f(x) = \frac{1}{x}$ on $[-1, 1]$ with $d = \frac{1}{2}$
[g] $f(x) = \frac{1}{x^2}$ on $[-2, 1]$ with $d = \frac{1}{9}$

[4] Which of the situations in [3] do NOT satisfy the conclusion of the Intermediate Value Theorem ?
[5] Why does each of the situations in [3] NOT contradict the Intermediate Value Theorem ?
Answer in terms of conditions and conclusions. Don’t just answer “because it’s a theorem”.
[6] Is it possible to find a function and a domain which contradict the Intermediate Value Theorem ?
Why or why not ?

Some additional problems for you to try after we cover section 4.2, based on Rolle's Theorem (Math 1A):

[1] What are the conditions of Rolle's Theorem ? HINT: There are 3 of them.

[2] What is the conclusion of Rolle's Theorem ?

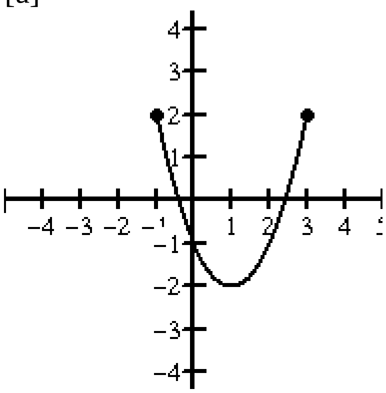
[3] Which of the functions below do NOT satisfy the conditions of Rolle's Theorem on the domain shown ? Name all conditions which are NOT satisfied. NOTE: [d], [g] have cusps.

[4] Which of these functions below do NOT satisfy the conclusion of Rolle's Theorem on the domain shown ?

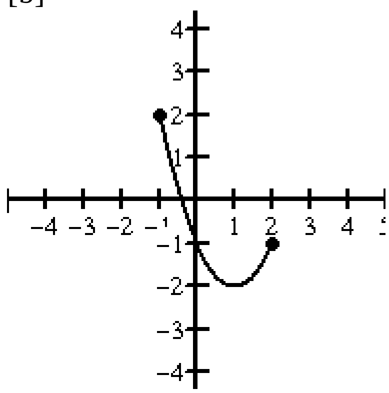
[5] Why does each of the functions below NOT contradict Rolle's Theorem ? Answer in terms of conditions and conclusions. Don't just answer "because it's a theorem".

[6] Is it possible to find a function and a domain which contradict Rolle's Theorem ? Why or why not ?

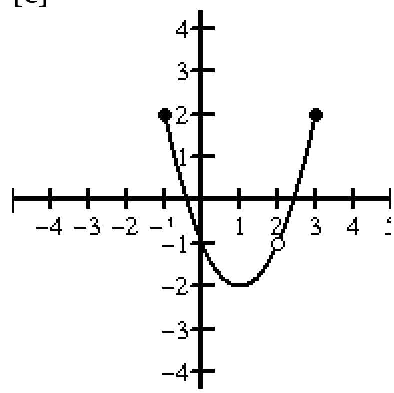
[a]



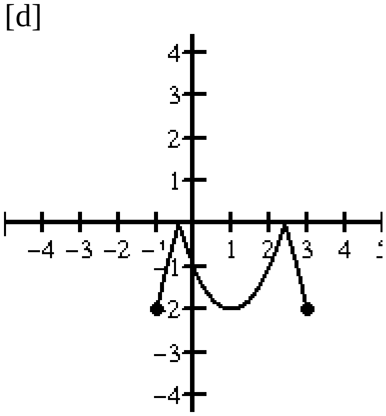
[b]



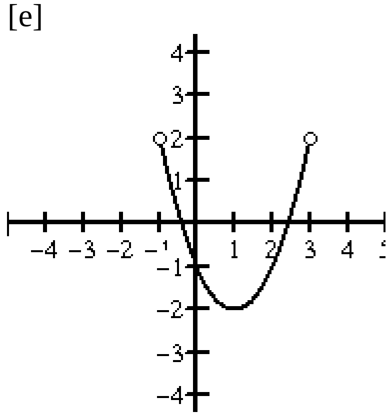
[c]



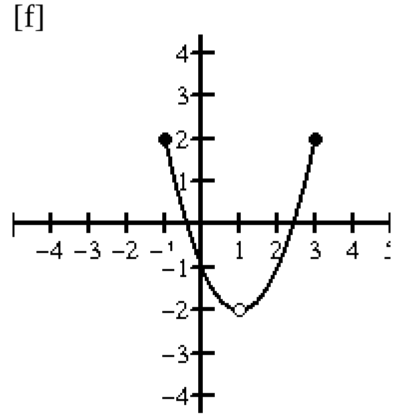
[d]



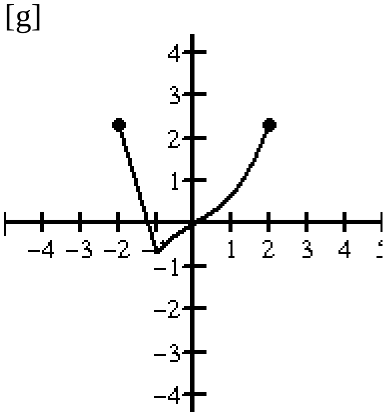
[e]



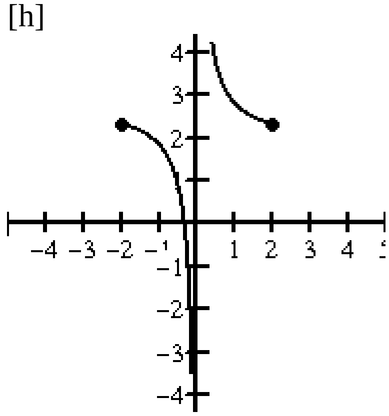
[f]



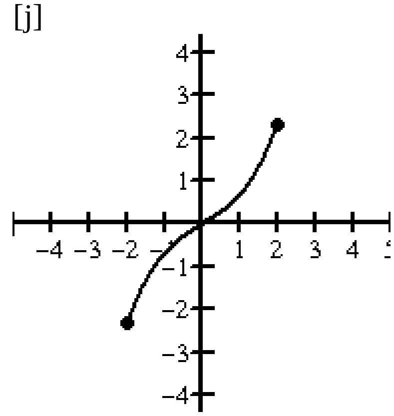
[g]



[h]



[j]



ANSWERS

[1] What are the conditions of the Intermediate Value Theorem ? HINT: There are 2 of them.

f is continuous on $[a, b]$
d is a number between $f(a)$ and $f(b)$

[2] What is the conclusion of the Intermediate Value Theorem ?

there is a number $c \in (a, b)$ such that $f(c) = d$

[3] Which of the following situations do NOT satisfy the conditions of the Intermediate Value Theorem ? Name all conditions which are NOT satisfied.

[a] satisfies the conditions
[b] $d = 5$ is not between $f(-3) = 9$ and $f(4) = 16$
[c] $f(x) = \frac{1}{x^2}$ is not continuous on $[-2, 1]$ at $x = 0$
[d] $f(x) = \frac{1}{x^2}$ is not continuous on $[-2, 1]$ at $x = 0$,
and $d = 4$ is not between $f(-2) = \frac{1}{4}$ and $f(1) = 1$
[e] $d = 25$ is not between $f(-3) = 9$ and $f(4) = 16$
[f] $f(x) = \frac{1}{x}$ is not continuous on $[-1, 1]$ at $x = 0$
[g] $f(x) = \frac{1}{x^2}$ is not continuous on $[-2, 1]$ at $x = 0$,
and $d = \frac{1}{9}$ is not between $f(-2) = \frac{1}{4}$ and $f(1) = 1$

[4] Which of the situations in [3] do NOT satisfy the conclusion of the Intermediate Value Theorem ?

[a] satisfies the conclusion since $f(3) = 9$ and $3 \in (1, 4)$
[b] satisfies the conclusion since $f(\sqrt{5}) = 5$ and $\sqrt{5} \in (-3, 4)$
[c] satisfies the conclusion since $f(\sqrt{2}) = \frac{1}{2}$ and $\sqrt{2} \in (-2, 1)$
[d] satisfies the conclusion since $f(\frac{1}{2}) = 4$ and $\frac{1}{2} \in (-2, 1)$
[e] does not satisfy the conclusion since $f(x) = 25$ only when $x = -5$ or $x = 5$,
and $-5 \notin (-3, 4)$ and $5 \notin (-3, 4)$
[f] does not satisfy the conclusion since $f(x) = \frac{1}{2}$ only when $x = 2$, and $2 \notin (-1, 1)$
[g] does not satisfy the conclusion since $f(x) = \frac{1}{9}$ only when $x = -3$ or $x = 3$,
and $-3 \notin (-2, 1)$ and $3 \notin (-2, 1)$

[5] Why does each of the situations in [3] NOT contradict the Intermediate Value Theorem ?
Answer in terms of conditions and conclusions. Don't just answer "because it's a theorem".

[a] satisfies both the conditions and the conclusion, so it does not contradict the theorem
[b]-[d] do not satisfy the conditions, so the theorem does not apply, so they do not contradict the

theorem (it is irrelevant that they satisfy the conclusion)
[e]-[g] do not satisfy the conditions, so the theorem does not apply, so they do not contradict the theorem (it is irrelevant that they do not satisfy the conclusion)

[6] Is it possible to find a function and a domain which contradict the Intermediate Value Theorem ? Why or why not ?

No. Because the Intermediate Value Theorem is a theorem, which means you cannot find any situation which contradicts it.

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[1] What are the conditions of Rolle's Theorem ? HINT: There are 3 of them.

f is continuous on $[a, b]$
f is differentiable on (a, b)
$f(a) = f(b)$

[2] What is the conclusion of Rolle's Theorem ?

there exists a $c \in (a, b)$ such that $f'(c) = 0$

[3] Which of the functions below do NOT satisfy the conditions of Rolle's Theorem on the domain shown ? Name all conditions which are NOT satisfied.

[a] satisfies the conditions
[b] $f(-1) \neq f(2)$
[c] not continuous on $[-1, 3]$ at $x = 2$
[d] not differentiable on $[-1, 3]$ at $x \approx -0.3, 2.3$
[e] not continuous on a closed interval
[f] not continuous on $[-1, 3]$ at $x = 1$
[g] not differentiable on $[-2, 2]$ at $x = -1$
[h] not continuous on $[-2, 2]$ at $x = 0$
[j] $f(-2) \neq f(2)$

[4] Which of these functions below do NOT satisfy the conclusion of Rolle's Theorem on the domain shown ?

[a]-[e] satisfy the conclusion since $f'(1) = 0$
[f]-[j] do not satisfy the conclusion since f' is never 0

[5] Why does each of the functions below NOT contradict Rolle's Theorem ?
Answer in terms of conditions and conclusions. Don't just answer "because it's a theorem".

[a] satisfies both the conditions and the conclusion, so it does not contradict the theorem
[b]-[e] do not satisfy the conditions, so the theorem does not apply, so they do not contradict the theorem (it is irrelevant that they satisfy the conclusion)
[f]-[j] do not satisfy the conditions, so the theorem does not apply, so they do not contradict the theorem (it is irrelevant that they do not satisfy the conclusion)

[6] Is it possible to find a function and a domain which contradict Rolle's Theorem ? Why or why not ?

No. Because Rolle's Theorem is a theorem, which means you cannot find any situation which contradicts it.